HORNSBY GIRLS' HIGH SCHOOL

2008
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using a black or blue pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown for every question
- Begin each question on a fresh sheet of paper

Total marks (120)

- Attempt Questions 1-10
- All questions are of equal value

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Total Marks – 120
Attempt Questions 1-10
All Questions are of equal value

Begin each question on a NEW SHEET of paper, writing your student number and question number at the top of the page. Extra paper is available.

Question 1 (12 marks) Use a SEPARATE sheet of paper. **Marks**

(a) Evaluate $\dfrac{2.3}{\sqrt[3]{2.45-1.09}}$ correct to 3 significant figures. **2**

(b) Factorise $y^3 + 125$. **2**

(c) Solve $\dfrac{x}{6} - \dfrac{x-1}{3} = 2$. **2**

(d) Find a primitive function of $2 + 3\sin x$. **2**

(e) Find the values of x for which $x^2 + 5x - 36 \geq 0$. **2**

(f) Write down the coordinates of the focus of the parabola $(x-2)^2 = -4(y+1)$. **2**

Question 2 (12 marks) Use a SEPARATE sheet of paper. **Marks**

(a) Solve $\sin\theta = \frac{1}{\sqrt{2}}$ for $0 \le x \le 2\pi$. **2**

(b) Differentiate with respect to x:

(i) $x\cos x$ **2**

(ii) $\frac{x^3}{3-x}$ **2**

(c) (i) Find $\int \frac{8x}{x^2-1}\,dx$. **2**

(ii) Evaluate $\int_0^{\frac{\pi}{8}} \sec^2 2x\,dx$. **2**

(d) Find the equation of the normal to $y = \log_e x^2$ at the point $(e, 2)$. **2**

Question 3 (12 marks) Use a SEPARATE sheet of paper. **Marks**

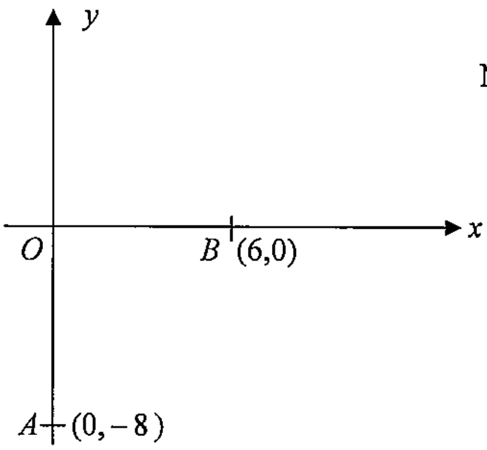
(a) Evaluate $\sum_{n=5}^{20}(3n-2)$. **2**

(b) The lengths of the sides of a triangle are 14cm, 16cm and 26cm

(i) Find the size of the angle opposite the smallest side **2**

(ii) Find the area of the triangle **1**

(c)



NOT TO SCALE

On the number plane above O, A and B are the points (0, 0), (0, – 8) and (6, 0) respectively.

(i) The point C is the point such that the mid-point of interval AC is (3, 0). Show that C has coordinates (6, 8). **1**

(ii) Show that the line AC has equation $3y = 8x - 24$. **2**

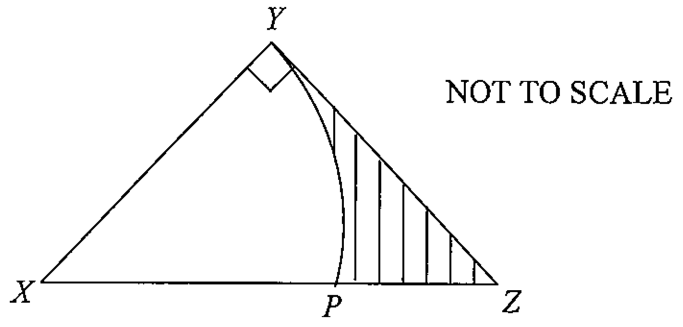
(iii) Show that $OABC$ is a parallelogram. Give reasons for your answer. **2**

(iv) Find the area of the parallelogram $OABC$ **1**

(v) Calculate the length of diagonal AC. **1**

Question 4 (12 marks) Use a SEPARATE sheet of paper. **Marks**

(a)



XYZ is an isosceles triangle right-angled at Y and $XY = 4$cm. An arc, centre X and radius 4cm is drawn to cut XZ at P.

(i) Copy this diagram onto your answer page and explain why $\angle YXZ = \frac{\pi}{4}$. **1**

(ii) Show that the area of the shaded portion YZP is $2(4-\pi)\,\text{cm}^2$. **2**

(iii) Find the perimeter of the shaded portion YZP. **2**

(b) A function $f(x)$ is defined by $f(x) = x^3 - 3x^2 - 9x + 1$.

(i) Find the coordinates of the turning points of the graph of $y = f(x)$, and determine their nature. **3**

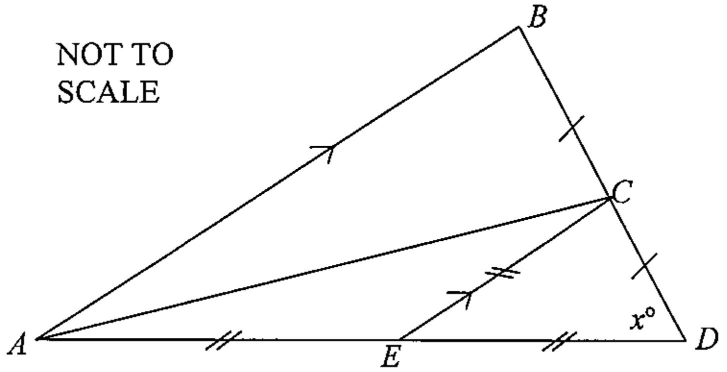
(ii) Hence sketch the graph of $y = f(x)$, showing the turning points and the y-intercept. **2**

(iii) Find the coordinates of any points of inflexion. **1**

(iv) For what values of x is the graph of $f(x)$ concave up? **1**

Question 5 (12 marks) Use a SEPARATE sheet of paper. **Marks**

(a) In the diagram, ABD is a triangle, the point C is the mid-point of BD and the point E is the mid-point of AD. Also, $AE = CE = DE$, $AB \parallel EC$ and $\angle EDC = x^0$.



(i) Show that $\angle ACB = 90^0$ **3**

(ii) Express $\angle BAD$ in terms of x^0 **1**

(b) Find the area bounded by $y = x^2 - 6x + 8$ and the x axis for $0 \le x \le 4$. **3**

(c) A school committee consists of five year 12 girls, six year 11 girls and two year 10 girls. Two girls are chosen at random from this committee to represent the school at a function. Find the probability that:

(i) they are both year 12 girls **1**

(ii) one year 12 girl and one year 11 girl is chosen **2**

(iii)at least one year 10 girl is chosen **2**

Question 6 (12 marks) Use a SEPARATE sheet of paper. **Marks**

(a) If $m+3m+5m+...+53m=81$, find the value of m. **3**

(b) The depth of water in the cross-section of a 4 metre wide creek was measured and recorded in the table below

Distance from one bank (m)	0	1	2	3	4
Depth (m)	0.4	0.8	1.5	1.3	0.3

(i) By applying Simpson's Rule, find the cross-sectional area of the creek at this point in square metres correct to 2 decimal places. **2**

(ii) The water at this point is flowing at a rate of 0.5 m/s. Calculate the volume of water which passes this point in one minute (answer to the nearest cubic metre). **1**

(c) The depth D, in metres, of a liquid stored in a vat at time t seconds is given by

$$D=\frac{t^2+1}{e^{2t}}, \quad t\geq 0$$

(i) What was the initial depth of the liquid in the vat? **1**

(ii) Find an expression for the rate at which the depth of the liquid changes **2**

(iii) Find the rate at which the depth of the liquid is changing when $t=2$. Hence, or otherwise, explain whether the depth is increasing or decreasing at this time. **2**

(iv) Are there any restrictions on the range of D? If so, state the range. **1**

Question 7 (12 marks) Use a SEPARATE sheet of paper. **Marks**

(a) Every year, starting on Lana's first birthday, her grandparents deposited \$100 for her in a special bank account at a rate of 9% p.a., compounded annually.

On her 21^{st} birthday, instead of depositing \$100, they deposited a lump sum of \$10 000 into the same account. After this, they stopped depositing money for Lana.

(i) How much did Lana have in her account immediately after the lump sum of \$10 000 was deposited? **3**

(ii) Lana left all the money in the bank at the same interest rate until her 26^{th} birthday. What was the balance of her account then? **2**

(iii) What single amount of money would Lana's grandparents have needed to invest on her first birthday so that she had the same amount of money on her 26^{th} birthday? **2**

(b) A particle is on a horizontal line. Initially, its velocity is 4 cm/s. It accelerates uniformly for 5 seconds until it is travelling with a velocity of 10 cm/s. It maintains this velocity for 6 seconds and then decelerates uniformly until it is at rest after a further 4 seconds.

(i) Show this information on a velocity - time graph. **2**

(ii) Find the distance travelled by the particle before it comes to rest. **2**

(iii) Find the average speed of the particle. **1**

Question 8 (12 marks) Use a SEPARATE sheet of paper. **Marks**

(a) The part of the curve $y = \frac{3}{x^2} + 1$ between $x = 2$ and $x = 3$ is rotated about the y-axis. Find the volume of the solid of revolution formed. **3**

(b) Let A and B be the fixed points $(2, 0)$ and $(-1, 0)$ let P be the variable point (x, y) such that $PA = 2PB$

(i) By finding the equation, deduce that the locus of P is a circle. **2**

(ii) Hence, or otherwise, find the centre and radius of this circle. **1**

(c) Jose borrowed \$250 000 at the beginning of 2008. The annual interest rate is 8%. At the end of each year, interest is calculated on the balance at the beginning of the year and added to that balance owing.
The debt is to be repaid by equal annual repayments of \$30 000, with the first payment being made at the end of 2008.

(i) Show that $A_2 = 250\,000(1.08)^2 - 30\,000(1 + 1.08)$ **1**

(ii) Show that $A_n = 375\,000 - 125\,000(1.08)^n$ **2**

(iii) In which year will Jose make the final repayment? **3**

Question 9 (12 marks) Use a SEPARATE sheet of paper. **Marks**

(a) (i) Explain why the series $1+(\sqrt{2}-1)+(\sqrt{2}-1)^2+\ldots$ has a limiting sum. **1**

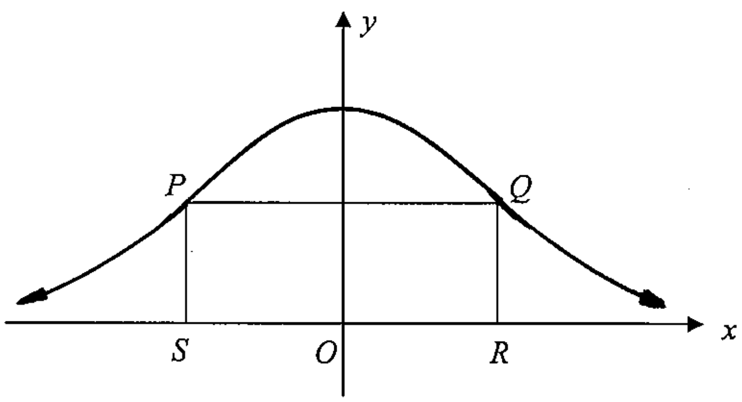
(ii) Calculate the limiting sum of this series **2**

(b) (i) On the same set of axes draw the graphs of $y=\cos 2x$ and $y=\sin x$ for the domain $0 \le x \le \pi$. **2**

(ii) Show that the curves intersect at $x=\dfrac{\pi}{6}$. **2**

(iii) If the x-value of the other point of intersection is $x=\dfrac{5\pi}{6}$, solve the inequality $\sin x \ge \cos 2x$ for $0 \le x \le \pi$. **1**

(c) The diagram shows a rectangle *PQRS* where *P* and *Q* are on the curve $y=e^{-x^2}$ and *R* and *S* are on the *x* axis. The point *O* is the origin and the lengths *OS* and *OR* are equal.



(i) Let the length *OR* be *x* units and show that the area of the rectangle *PQRS* is represented by the expression $A=2xe^{-x^2}$. **1**

(ii) Find the value of *x* for which *PQRS* has a maximum area. **3**

Question 10 (12 marks) Use a SEPARATE sheet of paper. **Marks**

(a) (i) Given that there are two tangents of the form $x - y = c$ to the circle $x^2 + y^2 = 2m^2$, prove that the x values of the points of contact are solutions of the equation $2x^2 - 2cx + (c^2 - 2m^2) = 0$. **1**

(ii) Hence prove that $c^2 - 4m^2 = 0$. **2**

(iii) If α, β are the roots of the equation $2x^2 - 2cx + (c^2 - 2m^2) = 0$ and $\alpha\beta = 9$, find the possible values of m and c. **2**

(iv) Hence find the equation of the circle and the equations of the two tangents. **2**

(b) Two councils in towns A and B have found that the population in the towns are given by:

$$P_A = 2000e^{-0.02t} \quad \text{for town A,}$$

$$P_B = 1000e^{0.03t} \quad \text{for town B,}$$

where t is the number of years which have elapsed since January 1st, 2000.

(i) Write down the annual growth rate for town B. **1**

(ii) Calculate the instantaneous rate at which the population of B will be increasing at the start of 2015. **2**

(iii) During which year will the population of B become larger than the population of A? **2**

End of paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

HGHS – MATHEMATICS TRIAL HSC 2008. SOLUTIONS (1)

Q1. a) $\doteqdot 2.075940\ldots$

$= 2.08$ (3 sig. figs)

b) $y^3+125 = (y+5)(y^2-5y+25)$

c) $\frac{x}{6} - \frac{x-1}{3} = 2$

$\frac{x}{6} - \frac{2(x-1)}{6} = \frac{12}{6}$

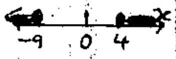
$x - 2x + 2 = 12$

$x = -10$

d) $\int 2+3\sin x \, dx = 2x - 3\cos x + C$

e) $x^2+5x-36 \geq 0$

$(x+9)(x-4) \geq 0$



$\therefore x \leq -9$ or $x \geq 4$

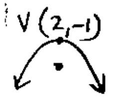
Test $x = 0$

$9 \times -4 \ngeq 0$

$-36 \ngeq 0$

f) $4a = -4$

$a = -1$



Focal length is 1 unit.

Vertex: $(2,-1)$

$\therefore$ Focus: $(2,-2)$

Q2. a) $\theta = \frac{\pi}{4}, \pi - \frac{\pi}{4}$ 1st, 2nd quadrants.

$= \frac{\pi}{4}, \frac{3\pi}{4}$

2b) (i) $\frac{d}{dx} x\cos x = x(-\sin x) + \cos x(1)$

$= \cos x - x\sin x$

(ii) $\frac{d}{dx} \frac{x^3}{3-x} = \frac{(3-x)(3x^2) - x^3(-1)}{(3-x)^2}$

$= \frac{9x^2 - 3x^3 + x^3}{(3-x)^2}$

$= \frac{9x^2 - 2x^3}{(3-x)^2}$

Q2 c) (i) $\int \frac{8x}{x^2-1} dx = 4\int \frac{2x}{x^2-1} dx$

$= 4\ln(x^2-1) + C$

(ii) $\int_0^{\frac{\pi}{8}} \sec^2 2x \, dx = \frac{1}{2}\left[\tan 2x\right]_0^{\frac{\pi}{8}}$

$= \frac{1}{2}\left\{\tan\frac{\pi}{4} - \tan 0\right\}$

$= \frac{1}{2} \times (1-0)$

$= \frac{1}{2}$

2d) $y = 2\log_e x$

$\frac{dy}{dx} = \frac{2}{x}$

When $x = e$, $\frac{dy}{dx} = \frac{2}{e}$

gradient of tangent $= \frac{2}{e}$

$\therefore$ gradient of normal $= -\frac{e}{2}$

Equation of normal given by:

$y - 2 = -\frac{e}{2}(x - e)$

$2y - 4 = -ex + e^2$

$ex + 2y - 4 - e^2 = 0$

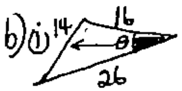
Q3. a) $\sum_{n=5}^{20} (3n-2) = (3\times5-2) + \ldots\ldots + (3\times20-2)$

$= 13 + \ldots\ldots + 58$ ($n = 16$ terms.)

$= \frac{16}{2}(13+58)$

$= 568$

3b) (i)



$\cos\theta = \frac{26^2+16^2-14^2}{2\times26\times16}$

$\theta \doteqdot 27.795788\ldots$

$\doteqdot 27°47'44.78''$

Smallest angle is 28° (to nearest degree)

(ii) $A = \frac{1}{2} \times 16 \times 26 \times \sin\theta$

$\doteqdot 96.994879$ or using 28°, $\doteqdot 96.65\ldots$

area is 97 cm² (to nearest whole number cm²)

Q3 (cont'd)

c) (i) Let C be (x_2, y_2), Midpoint of AC is $(3,0)$

$3 = \frac{0+x_2}{2}$, $0 = \frac{-8+y_2}{2}$

$x_2 = 6$ $\quad y_2 = 8$ $\quad \therefore$ C is $(6,8)$

(ii) Equation of AC: $\frac{y-8}{x-6} = \frac{-8-8}{0-6}$

$y - 8 = \frac{16}{6}(x-6)$ [$\frac{8}{3}$]

$3y - 24 = 8x - 48$

$3y = 8x - 24$

(iii) Midpoint of OB is $(3,0)$ (given)

" " AC is also $(3,0)$ (given)

$\therefore$ OABC is a parallelogram (diagonals bisect each other)

OR $m_{OC} = \frac{8-0}{6-0} = \frac{4}{3}$, $m_{AB} = \frac{0+8}{6-0} = \frac{4}{3}$

$\therefore OC \parallel AB$

Eqn of OA is $x = 0$ and the Eqn of CB is $x = 6$ } both vertical lines

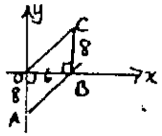
$\therefore OA \parallel CB$

$\therefore$ OABC is a parallelogram (both pairs of sides parallel).

(iv) $A = 2 \times \frac{1}{2} \times 6 \times 8$

$= 48$

Area is 48 units²



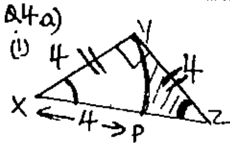
(v) $d_{AC} = \sqrt{6^2 + (8+8)^2}$

$= \sqrt{292}$

$= 2\sqrt{73}$ units.

Q4 a)

(i)



$\triangle YXZ$ is isosceles (given)

$\angle Y$ is a right angle (given)

$\therefore \angle X = \angle Z$ (other two angles of $\triangle$)

$= \frac{180° - 90°}{2}$

$= 45°$

$\therefore \angle YXZ = \frac{\pi}{4}$

Q4 a) (ii) Area of YZP shaded portion = Area of $\triangle XYZ$ − Area of Sector XYP

$A = \frac{1}{2}\times4\times4 - \frac{1}{2}4^2\left(\frac{\pi}{4}\right)$

$= 8 - 2\pi$

$\therefore$ Area is $2(4-\pi)$ cm²

(iii) Perimeter, $P = 4 + 4\times\frac{\pi}{4} + \left(\sqrt{4^2+4^2} - 4\right)$ [YZ + Arc YP + PZ]

$= 4 + \pi + 4\sqrt{2} - 4$

Perimeter is $(\pi + 4\sqrt{2})$ cm.

b) (i) $f(x) = x^3 - 3x^2 - 9x + 1$

$f'(x) = 3x^2 - 6x - 9$

$f''(x) = 6x - 6$

Turning points occur when $f'(x) = 0$

ie $3(x^2 - 2x - 3) = 0$

$3(x-3)(x+1) = 0$

$x = 3$ or $x = -1$

When $x = 3$, $f''(3) = 6\times3 - 6$

$= 12 > 0$ ∪

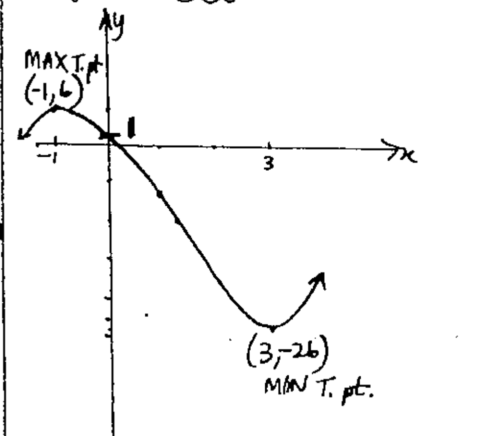
$\therefore$ Min. T. point at $(3,-26)$

When $x = -1$, $f''(-1) = 6\times-1-6$

$= -12 < 0$ ∩

$\therefore$ Max. T. point at $(-1,6)$

(ii) y-intercept is 1



Q4(cont'd)

(iii) Possible inflexion points occur when $f''(x) = 0$ i.e. $6x - 6 = 0$, $6(x-1) = 0$, $x = 1$

Test for change in concavity.

For x a little less than 1, say $x = 0.9$: $f''(0.9) = -0.6 < 0$

For x a little more than 1, say $x = 1.1$: $f''(1.1) = 0.6 > 0$

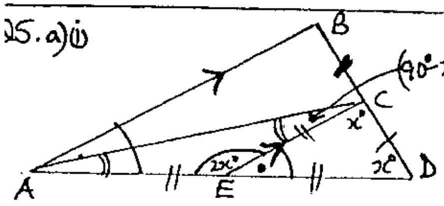
$\therefore$ Change in concavity, hence Inflexion Point at $(1, -10)$

(iv) Concave Up for $f''(x) > 0$

$6x - 6 > 0$

$x > 1$

Q5. a)(i)



In ΔECD, $\angle ECD = x^\circ$ (angles opposite equal sides in isosceles Δ.)

$\therefore \angle CEA = 2x^\circ$ (exterior angle of ΔECD)

in ΔAEC, $\angle ECA = \angle EAC$ (angles opposite equal sides in isosceles Δ)

$= \frac{180^\circ - 2x^\circ}{2}$ (angle sum of ΔAEC)

$= 90^\circ - x^\circ$

$\therefore \angle ACB = 180^\circ - (\angle AEC + \angle ECD)$ (straight angle)

$= 180^\circ - (90^\circ - x^\circ + x^\circ)$

$= 90^\circ$.

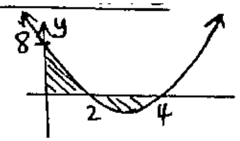
(ii) $\angle BAD = \angle CED$ (corresponding angles, $AB \parallel EC$)

$\therefore \angle BAD = 180^\circ - (2x^\circ)$ (angle sum of ΔCED or straight angle)

$= 180^\circ - 2x^\circ$

Q5 b) $y = x^2 - 6x + 8$

$= (x-4)(x-2)$



$$A = \int_0^2 y \, dx + \left| \int_2^4 y \, dx \right|$$

$$= \left[\frac{x^3}{3} - \frac{6x^2}{2} + 8x\right]_0^2 + \left|\left[\frac{x^3}{3} - 3x^2 + 8x\right]_2^4\right|$$

$$= \left(\frac{8}{3} - 12 + 16\right) - 0 + \left|\left(\frac{64}{3} - 48 + 32\right) - \left(\frac{8}{3} - 12 + 16\right)\right|$$

Area is 8 units²

5c) (i) P(Both Year 12) $= \frac{5}{13} \times \frac{4}{12}$

$= \frac{5}{39}$

(ii) P(one yr 12, one yr 11) $= P(12,11) + P(11,12)$

$= \frac{5}{13} \times \frac{6}{12} + \frac{6}{13} \times \frac{5}{12}$

$= \frac{5}{13}$

(iii) P(at least 1 Yr 10) $= 1 - P(\text{no year } 10)$

$= 1 - \left(\frac{11}{13} \times \frac{10}{12}\right)$

$= 1 - \frac{55}{78}$

$= \frac{23}{78}$

Q6 a) Arithmetic Series $a = m$, $d = 2m$, $l = 53m$.

$53m = m + (n-1)2m$.

$52m = (n-1)2m$

$n - 1 = 26$

$n = 27$

$\therefore 81 = \frac{27}{2}(m + 53m)$

$54m = 6$

$m = \frac{1}{9}$

Q6 (cont'd) b)

(i) $A \doteqdot \frac{2-0}{6}\left[0.4 + 4 \times 0.8 + 1.5\right] + \frac{4-2}{6}\left[1.5 + 4 \times 1.3 + 0.3\right]$

$= 4.0\dot{3}$

Area is 4.03 m^2 (correct to 2 dec. pl.)

(ii) $V = 4.03 \times 0.5 \times 60$

$= 120.9$

$\therefore$ Volume is 121 m^3 (correct to nearest m^3)

6c) (i) $t = 0$, $D = \frac{0+1}{e^0}$

$= 1$

Initial depth is 1 metre.

(ii) $\frac{dD}{dt} = \frac{e^{2t} \times 2t - (t^2+1)(2e^{2t})}{(e^{2t})^2}$

$= \frac{2e^{2t}(t - t^2 - 1)}{(e^{2t})^2}$

Rate of change is $\frac{2(t - t^2 - 1)}{e^{2t}}$ m/s.

(iii) When $t = 2$, $\frac{dD}{dt} = \frac{2(2 - 2^2 - 1)}{e^4}$

Rate is $\frac{-6}{e^4}$ m/s.

$\therefore$ Depth is decreasing since $\frac{dD}{dt} < 0$ when $t = 2$.

(iv) Depth is initially 1 metre and since $t^2 \neq -1 \therefore D \neq 0$

Also $e^{2t} > 0$, $t^2 + 1 \geq 1 \therefore D > 0$

$e^{2t} \geq t^2 + 1 \therefore \frac{e^{2t}}{t^2+1} \leq 1$

Range: $0 < D \leq 1$

Q7. a)

(i) 1st \$100 amounts to \$100$(1.09)^{20}$

2nd \$100 " " \$100$(1.09)^{19}$

3rd \$100 " " \$100$(1.09)^{18}$

⋮

20th \$100 " " \$100$(1.09)^1$

on 21st \$10000 is deposited.

Total $= \$100\,(1.09 + 1.09^2 + 1.09^3 + \ldots + 1.09^{20}) + \$10000$

geometric series $a = 1.09$ $r = 1.09$ $n = 20$

$= \$100 \times \frac{1.09(1 - 1.09^{20})}{1 - 1.09} + \$10000$

$= \$15576.45$ (to nearest cent)

(ii) Amount on 26th Birthday $= \$15576.45(1.09)^5$

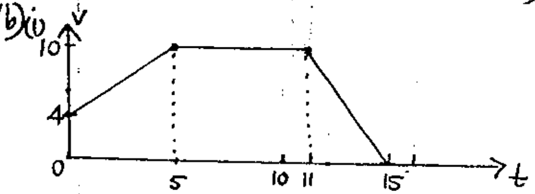
$= \$23966.30$ (to nearest cent)

(iii) $23966.30 = P(1.09)^{25}$

$P = \frac{23966.30}{1.09^{25}}$

Amount is \$2779.32 (to nearest cent)

7b)(i)



(ii) Distance travelled, $d = \int v \cdot dt$ (area under graph)

$= \frac{5}{2}(4 + 10) + 6 \times 10 + \frac{1}{2} \times 4 \times 10$

Distance travelled is 115 cm

(iii) Average Speed $= \frac{\text{Distance travelled}}{\text{Time Taken}}$

$= \frac{115}{15}$ cm/s

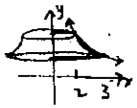
$= 7\frac{2}{3}$ or $7.\dot{6}$ cm/s.

Q8a) $y = \frac{3}{x^2} + 1$ When $x = 2$, $y = \frac{7}{4}$; $x = 3$, $y = \frac{4}{3}$

$V = \pi \int_{\frac{4}{3}}^{\frac{7}{4}} x^2 \, dy$

$\left\{ y - 1 = \frac{3}{x^2} \quad x^2 = \frac{3}{y-1} \right.$

$= \pi \int_{\frac{4}{3}}^{\frac{7}{4}} \frac{3}{y-1} \, dy$



$= 3\pi \left[\ln(y-1) \right]_{\frac{4}{3}}^{\frac{7}{4}}$

$= 3\pi \left[\ln\left(\frac{3}{4}\right) - \ln\left(\frac{1}{3}\right) \right]$

$= 3\pi \ \ln\left(\frac{3}{4} \div \frac{1}{3}\right)$

Volume is $3\pi \ln \frac{9}{4}$ units^3.

8b)(i) $A(2,0)$ $B(-1,0)$ $P(x,y)$

$PA = 2PB$

$PA^2 = 4PB^2$

$(x-2)^2 + (y-0)^2 = 4\left[(x+1)^2 + (y-0)^2\right]$

$x^2 - 4x + 4 + y^2 = 4(x^2 + 2x + 1) + 4y^2$

$x^2 - 4x + 4 + y^2 = 4x^2 + 8x + 4 + 4y^2$

$3x^2 + 12x + 3y^2 = 0$

$x^2 + 4x + y^2 = 0$

$(x+2)^2 + y^2 = 4$ } Eqn of a Circle

(ii) Centre is $(-2,0)$ Radius is 2 units

8c) (i) $A_1 = 1.08 \times 250\,000 - 30\,000$

$A_2 = A_1 \times 1.08 - 30\,000$

$= 1.08^2 \times 250\,000 - 1.08 \times 30\,000 - 30\,000$

$= 250000(1.08)^2 - 30000(1 + 1.08)$

(ii) $A_3 = 250000(1.08)^3 - 30000(1.08 + 1.08^2) - 30000$

$= 250000(1.08)^3 - 30000(1 + 1.08 + 1.08^2)$

⋮

8c (ii) (contd)

$A_n = 250000(1.08)^n - 30000(1 + 1.08 + 1.08^2 + \ldots + 1.08^{n-1})$

geometric series with $a = 1$, $r = 1.08$, $n = n$

$= 250000(1.08)^n - 30000\left(\frac{1 - 1.08^n}{1 - 1.08}\right)$

$= 250000(1.08)^n + 375000(1 - 1.08^n)$

$= 250000(1.08)^n - 375000(1.08)^n + 375000$

$= 375000 - 125000(1.08)^n$

(iii) When final repayment made $A_n = 0$

$\therefore 125000(1.08)^n = 375000$

$1.08^n = \frac{375000}{125000}$

$= 3$

$n \log 1.08 = \log 3$

$n = \frac{\log 3}{\log 1.08}$

$\doteqdot 14.27\ldots$

$\therefore$ Final repayment made in 15th year which is 2023.

Q9 a)(i) A Limiting sum exists if $|r| < 1$

$r = \frac{\sqrt{2} - 1}{1} = \frac{(\sqrt{2} - 1)^2}{\sqrt{2} - 1}$

$= \sqrt{2} - 1$

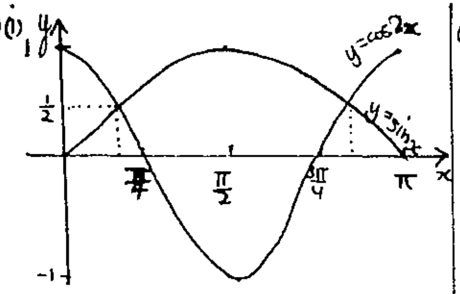
$\doteqdot 0.414\ldots \quad \therefore |r| < 1$

(ii) $S_\infty = \frac{a}{1 - r}$

$= \frac{1}{1 - (\sqrt{2} - 1)}$

$= \frac{1}{2 - \sqrt{2}}$ or $\frac{2 + \sqrt{2}}{2}$

Q9 b)(i)



(ii) $\sin \frac{\pi}{6} = \frac{1}{2}$

$\cos 2\left(\frac{\pi}{6}\right) = \cos \frac{\pi}{3} = \frac{1}{2}$

} $\therefore$ Curve intersect at $x = \frac{\pi}{6}$

(iii) $\sin x \geq \cos 2x$ for $0 \leq x \leq \pi$

for $\frac{\pi}{6} \leq x \leq \frac{5\pi}{6}$

9c) (i) $R(x,0)$ $Q(x, e^{-x^2})$ $S(-x,0)$

$A = b \times h$

$\therefore A_{PQRS} = 2x \times e^{-x^2}$

(ii) $\frac{dA}{dx} = 2x(-2x \cdot e^{-x^2}) + e^{-x^2} \times 2$

$= 2e^{-x^2} - 4x^2 e^{-x^2}$

$= 2e^{-x^2}(1 - 2x^2)$

For Max/min area $\frac{dA}{dx} = 0$

i.e. $2e^{-x^2}(1 - 2x^2) = 0$

cannot be 0, No solution. $x = \pm \frac{1}{\sqrt{2}}$

But by symmetry (of graph) both these values give the same area.

Test $x = \frac{1}{\sqrt{2}}$,

x	0	$\frac{1}{\sqrt{2}}$	1
$\frac{dA}{dx}$	2	0	$-\frac{2}{e}$

+ 0 − MAX.

$\therefore$ MAXIMUM Area occurs when $x = \pm \frac{1}{\sqrt{2}}$

Q10 a)(i) $x - y = c \implies y = x - c$ — (1)

$x^2 + y^2 = 2m^2$ — (2)

Substituting equation (1) into (2)

$x^2 + (x - c)^2 = 2m^2$

$x^2 + x^2 - 2cx + c^2 = 2m^2$

$2x^2 - 2cx + (c^2 - 2m^2) = 0.$

(ii) A line is a tangent to a circle if it touches it (one point of contact)

$\therefore \Delta = 0$

i.e. $0 = (-2c)^2 - 4 \times 2 \times (c^2 - 2m^2)$

$= 4c^2 - 8(c^2 - 2m^2)$

$= 16m^2 - 4c^2$

$0 = 4(4m^2 - c^2)$

i.e. $c^2 - 4m^2 = 0$.

(iii) $\alpha\beta = 9$

$\frac{c^2 - 2m^2}{2} = 9$

$c^2 - 2m^2 = 18$ and $c^2 - 4m^2 = 0$ from (ii)

$c^2 = 4m^2$

Substituting $c^2 = 4m^2$ gives: $4m^2 - 2m^2 = 18$

$2m^2 = 18$

$m^2 = 9$

$m = \pm 3$

$\therefore c^2 = 4 \times 9$, since $m^2 = 9$

$= 36$

$c = \pm 6$

(iv) Circle equation: $x^2 + y^2 = 2 \times 9$

i.e. $x^2 + y^2 = 18$

Eqns of tangents: $x - y = \pm 6$

i.e. $x - y = 6$ or $x - y + 6 = 0$

$x - y - 6 = 0$

Q10b) (i) $\frac{dP_B}{dt} = 1000 \times 0.03\, e^{0.03t} = 0.03\, P_B$. ∴ annual growth rate for Town B is <u>0.03 people/year</u>

(ii) $\frac{dP_B}{dt} = 30\, e^{0.03t}$

2015, $t = 15$

$$\frac{dP_B}{dt} = 30e^{0.03(15)}$$

$$\doteqdot 47.049366\ldots$$

Instanteous rate of increase is <u>47 people/year</u> (to nearest whole person).

(iii) $1000\, e^{0.03t} > 2000\, e^{-0.02t}$

$$\frac{e^{0.03t}}{e^{-0.02t}} > \frac{2000}{1000}$$

$$e^{0.05t} > 2$$

$$\ln(e^{0.05t}) > \ln 2$$

$$0.05t > \ln 2$$

$$t > \frac{\ln 2}{0.05}$$

$$> 13.8629\ldots$$

∴ During 14th year after 2000 i.e. <u>2014</u>. population of B becomes larger than population of A.